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HW6 , Math 531, Spring 2014

Ayman Badawi

QUESTION 1. (i) Let R be a commutative ring with $1 \neq 0$. Then (Hint: for this question, try to use class notes)

a. Let P be a prime ideal of R. We know that $P[X]$ is an ideal of $R[X]$. Prove that $P[X]$ is a prime ideal of $R[X]$.

b. Let M be a maximal ideal of R. We know that $M[X]$ is an ideal of R. Prove that $M[X]$ is a prime ideal of $R[X]$ but never a maximal ideal of $R[X]$.

(ii) Give me an example of a commutative ring with identity, say R, such that R has two prime ideals P, Q that are not co-prime such that neither $P \subset Q$ nor $Q \subset P$.

(iii) Let R be a commutative ring with $1 \neq 0$. An element $e \in R$ is called idempotent if $e^2 = e$.

a. Prove that if e is an idempotent of R, then $(1 - e)$ is an idempotent of R.

b. Prove that if e is an idempotent of R, then $(1 - 2e) \in U(R)$.

c. Find all idempotent elements of $Z_{30} = Z/30Z$. [Hint: You may find them by try and error, but I recommend that you find the idempotents of Z_6, then use Chinese Remainder Theorem]

d. Let e be an idempotent of a commutative ring R with $1 \neq 0$. Then it is clear that $I = eR, J = (1 - e)R$ are ideals of R. Prove that R is ring-isomorphic to $R/I \times R/J$.

(iv) (Computational): We know that $X^2 + X + 2$ has no roots in Z. Let $I = (X^2 + X + 2)Z[X]$ is an ideal of $Z[X]$, and let $A = Z[X]/I$. Find all roots of $P(Y) = (1 + I)Y^2 + (1 + I)Y + (2 + I) \in A[Y]$ over A. Note that over A sometimes we write $P(Y)$ as $Y^2 + Y + 2$]

(v) (computational): We know that $I = 6Z[X]$ and $J = 11Z[X]$ are ideals of $Z[X]$. Let $A = Z[X]/I$ and $B = Z[X]/J$. Find a polynomial $P(X)$ in $Z[X]$ such that $P(X) + I = (X + 3) + I$ (in A) and $P(X) + J = (X^3 + 7X^2 - 2X + 9) + J$ (in B).

Faculty information

Ayman Badawi, Department of Mathematics & Statistics, American University of Sharjah, P.O. Box 26666, Sharjah, United Arab Emirates.
E-mail: `abadawi@aus.edu, www.ayman-badawi.com`